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THE PROPERTY OF INNER SYMMETRY IN OPTIMUM MULTILAYER STRUCTURES UNDER THE ACTION OF ELASTIC WAVES†

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The general properties of laminar, linearly elastic, locally isotropic bodies which are energetically optimum for a specified inclined incidence of a plane wave are investigated. It is shown that, in the case of special supplementary conditions of the type that the frequencies of the maximum attainable value of the energy characteristic should be present in the required spectrum or that the angle of incidence is sufficiently small, the problem reduces to an alternate configuration of no more than two of the given set of layers which are identical both in their physical properties and in their width. These properties were established earlier for the propagation of electromagnetic and purely longitudinal waves. © 2002 Elsevier Science Ltd. All rights reserved.

The effective solution of the problem of investigating the limiting possibilities of multilayer structures in processes involving the control of the power of wave processes of a diverse physical nature assumes that it is possible to carry out an effective exhaustive search of all permissible versions of the constructions, the number of which is exceedingly large. However, within the framework of existing approaches, it is not possible either to estimate objectively to what extent the possibilities of the structures which are created differ from the maximum attainable possibilities or to construct structures efficiently which realize these limiting possibilities. We therefore formulated the problem of devising methods for a multilateral investigation of the limiting possibilities of composite structures, and we proposed a hypothesis concerning the existence of general relationships which are characteristic of structures that realize the limiting possibilities.

The design of structures with unique properties involves investigating their limiting possibilities, which correspond to that limiting level which can be attained by controlling the structure of the system. In multi-extremum problems, to which wave problems of synthesis belong, the possibilities for predicting the behaviour of the objective function are limited. A local prediction is insufficient for constructing efficient procedures for searching for solutions.

In cases of the inclined incidence of electromagnetic waves on a system of magnetodielectric layers as well as of the oblique incidence of acoustic waves on a system of layers in which shear waves do not propagate, qualitative rules for the structure of the optimum solution have been established. In particular, it was shown in [1]‡ that the relation between the parameters in structures, which realize the limiting possibilities with respect to the attainment of a specified set of properties, possesses a definite inner symmetry, that enables one effectively to separate out the complete set of versions which realize the limiting possibilities in the case of such problems. In this case, the same type of symmetry is characteristic of the relation between the parameters in the optimum structures under the action of both electromagnetic and acoustic waves.

The question arises as to whether the established type of symmetry in the relation between the parameters in the optimum structure is preserved on transferring to wave problems of synthesis which are described by more complex models, in particular, when the possibility arises of a transformation of different types of waves at the interfaces. This paper is concerned with this question.

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1. FORMULATION OF THE PROBLEM OF OPTIMUM SYNTHESIS

Suppose it is required to construct a multilayer structure which will provide the limiting possibilities with respect to the attenuation of elastic waves in a specified frequency band $[\omega_{\min}, \omega_{\max}]$. We shall assume that the half-spaces bordering the system of layers are ideal fluids.

The propagation of an elastic wave in a system of elastic layers is described by the system of dynamical equations of the theory of elasticity

$$\rho_s \frac{\partial^2 \mathbf{u}_s}{\partial t^2} = \mu_s \Delta \mathbf{u}_s + (\lambda_s + 2\mu_s) \operatorname{grad} \operatorname{div} \mathbf{u}_s, \quad s = 1, \dots, N \tag{1.1}$$

Here, $\mathbf{u}_s(x, y, z, t)$ is the vector of the displacements of the particles in the sth medium, ρ_s is the density of the sth layer, λ_s, μ_s are the Lamé parameters of the sth layer and N is the number of layers.

The vector field of the displacements can be represented in the form of a superposition of two fields [2]

$$\mathbf{u}_s = \operatorname{grad} \Phi_s + \operatorname{rot} \mathbf{P}_s \tag{1.2}$$

Here Φ_s and $\mathbf{P}_s$ are the scalar and vector potentials of the wave field. A plane wave of general form can be represented in the form of a superposition of plane harmonic waves

$$\left\| \begin{matrix} \Phi_s(x, y, z, t) \\ \mathbf{P}_s(x, y, z, t) \end{matrix} \right\| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left\| \begin{matrix} f_s^+(z, \omega) \\ f_s^-(z, \omega) \end{matrix} \right\| \exp(i\Delta_0 x + i\Delta_0 y - i\omega t) d\omega, \quad s = 1, \dots, N \tag{1.3}$$

$$\Delta_0 = k_0 \sin \vartheta_0, \quad k_0 = \omega / c_0$$

Here k_0 is the wave number of the incident wave, c_0 is the propagation velocity of the wave in the half-space from which the wave comes and ϑ_0 is the angle of incidence of the wave.

The problem of the propagation of an elastic wave in a system of two elastic layers then reduces to solving the following boundary-value problem in the spectral densities of the scalar and vector potentials

$$\begin{gathered}
\frac{\partial^2 f_s^+(z, \omega)}{\partial z^2} + (k_s^2(\omega) - \Delta_0^2(\omega)) f_s^+(z, \omega) = 0 \\
\frac{\partial^2 f_s^-(z, \omega)}{\partial z^2} + (\gamma_s^2(\omega) - \Delta_0^2(\omega)) f_s^-(z, \omega) = 0 \\
b_{s-1} \leqslant z \leqslant b_s; \quad s = 1, \dots, N \\
f_s^\pm(b_{s-1}, \omega) = (\Phi_s + Q_s) f_{s-1}^\pm(b_{s-1}, \omega) \pm i \frac{Q_s}{\Delta_0} \frac{\partial f_{s-1}^\mp(b_{s-1}, \omega)}{\partial z} \\
\frac{\partial f_s^\pm(b_{s-1}, \omega)}{\partial z} = (\Phi_s - Q_s) \frac{\partial f_{s-1}^\pm(b_{s-1}, \omega)}{\partial z} \pm i\Delta_0 (1 - \Phi_s - Q_s) f_{s-1}^\mp(b_{s-1}, \omega) \\
s = 2, \dots, N \\
\frac{\partial f_0^+(0, \omega)}{\partial z} = -i \cos \vartheta_0 \left[k_0 f_0^+(0, \omega) + \frac{2}{k_0 \rho_0 c_0^2} \right], \quad f_0^-(0, \omega) = 0 \\
\frac{\partial f_{N+1}^+(l, \omega)}{\partial z} = i k_{N+1} \cos \vartheta_{N+1} f_{N+1}^+(l, \omega), \quad f_{N+1}^-(l, \omega) = 0
\end{gathered} \tag{1.4}$$

where

$$\Phi_s = \frac{\mu_{s-1} \gamma_{s-1}^2}{\mu_s \gamma_s^2}, \quad Q_s = 2\Delta_0^2 \frac{\mu_s - \mu_{s-1}}{\mu_s \gamma_s^2}; \quad s = 2, \dots, N$$

Here $k_s = \omega / c_s$ is the wave number of the longitudinal wave in the sth layer, c_s is the propagation velocity of the longitudinal wave in the sth layer, $\gamma_s = \omega / d_s$ is the wave number of the shear wave in it and

b_s ($s = 1, \ldots, N$) are the coordinates of the interfaces of the layers with different physical properties. The conditions at the interfaces are a consequence of the rigid adhesion of the layers when the normal and tangential components of the displacements as well as the normal and shear stresses must be continuous at the interfaces.

Suppose a discrete set of materials, which can participate in the design, is specified. The physical properties of each of these materials will be related by certain functional dependences. For instance, the propagation velocities of longitudinal and transverse waves in the material will be related to its density ρ: $c = c(\rho)$, $d = d(\rho)$. We now introduce the set of densities of the materials from the admissible set

$$\Lambda = \{\rho_{\min} = \rho^1 < \rho^2 < \rho^3 < \ldots < \rho^m = \rho_{\max}\}$$

Over this set, the quantities c and d are function of the discrete argument $\rho \in \Lambda$.

It is therefore necessary to select the physical properties of the materials of the layers ρ_s ($s = 1, \ldots, N$), the thicknesses of the layers $\Delta_s = b_s - b_{s-1}$($s = 1, \ldots, N$), the number of layers N and, also, the order in which they are arranged in order that the dependence of the energy transmission coefficient of the structure being designed should be closest to the required dependence $\tilde{T}(\omega)$. The problem reduces to minimizing the quality criterion

$$J = \int_{\omega_{\min}}^{\omega_{\max}} \tau(\omega)[T(\omega) - \tilde{T}(\omega)]^2 d\omega, \quad T(\omega) = \frac{c_0\rho_0 \cos\vartheta_{N+1}}{c_{N+1}\rho_{N+1}\cos\vartheta_0} |f_{N+1}(l, \omega)|^2 \tag{1.5}$$

for the solutions of system (1.4). Here, $\tau(\omega)(0 \leqslant \tau(\omega) \leqslant 1)$ is a weighting function.

Problem (1.4), (1.5) belongs to a number of optimal control problems of composite systems of the combinatorial type which have been studied previously [1]; the necessary optimality conditions for composite systems with such a structure have already been formulated.

2. THE NECESSARY CONDITIONS OF OPTIMALITY

The necessary optimality conditions, which are an extension of Pontryagin's maximum principle to problems of the optimal control of composite systems [1], can be formulated for optimal control problem (1.4), (1.5). Here, the functions which, for optimal control problem (1.4), (1.5), can be represented in the form [1]

$$R_s(f_s^+, f_s^-, p_s, q_s; \rho)|_z = \int_{\omega_{\min}}^{\omega_{\max}} \sum_{i=1}^{8} \alpha_s^i(z, \omega) G_s^i(\omega; \rho) d\omega \tag{2.1}$$

$$b_{s-1} \leqslant z \leqslant b_s, \quad s = 1, \ldots, N$$

are an analogue of Hamilton's functions. (Note that, on the right-hand side of formula (2.1), all the arguments of the function R_s are determined at the point z.)

The functions $\alpha_s^i = \alpha_s^i(z, \omega)(i = 1, \ldots, 8)$ are expressed in terms of the solution

$$f_s^+(z, \omega), \ f_s^-(z, \omega), \ b_{s-1} \leqslant z \leqslant b_s, \ s = 1, \ldots, N$$

of the initial system (1.4) and the solution

$$p_s(z, \omega), \ q_s(z, \omega), \ b_{s-1} \leqslant z \leqslant b_s, \ s = 1, \ldots, N$$

of the system which is conjugate to (1.4) [1]

$$\alpha_s^1 = \operatorname{Im}\frac{\partial p_s}{\partial z} f_s^+, \ \alpha_s^2 = \operatorname{Im}\frac{\partial f_s^+}{\partial z} p_s, \ \alpha_s^3 = \operatorname{Im}\frac{\partial q_s}{\partial z} f_s^-, \ \alpha_s^4 = \operatorname{Im}\frac{\partial f_s^-}{\partial z} q_s$$

$$\alpha_s^5 = \operatorname{Re} f_s^+ q_s, \ \alpha_s^6 = \operatorname{Re}\frac{\partial f_s^+}{\partial z}\frac{\partial q_s}{\partial z}, \ \alpha_s^7 = \operatorname{Re} p_s f_s^-, \ \alpha_s^8 = \operatorname{Re}\frac{\partial p_s}{\partial z}\frac{\partial f_s^-}{\partial z} \tag{2.2}$$

The functions $G_s^i = G_s^i(\omega; \rho)$ have the form

$$G_s^1 = \frac{\omega}{1-2\xi_s(\omega)}\{-(n_s(\rho)\gamma_s^2(\omega)-\Delta_0^2(\omega))+\Delta_0^2(\omega)(1-\tilde{\rho})+$$
$$+4\Delta_0^2(\omega)(n_s(\rho)-\nu(\rho)+4\Delta_0^4(\omega)h_s(\rho))\}$$
$$G_s^2 = \frac{\omega\eta_s(\omega)}{\xi_s(\omega)}\left\{\tilde{\rho}+4\xi_s(\omega)\left(\frac{1}{m_s(\rho)}-1\right)\right\}.$$
$$G_s^3 = \omega\left\{\frac{\gamma_s^2(\omega)}{m_s(\rho)}(1-2\xi_s(\omega))+\tilde{\rho}\frac{\Delta_0^2(\omega)}{1-2\xi_s(\omega)}+2\Delta_0^2(\omega)\right\}$$
$$G_s^4 = \frac{\omega(1-2\xi_s)}{1-\xi_s(\omega)}\{-\tilde{\rho}\gamma_s^2(\omega)+4\Delta_0^2(\omega)h_s(\rho)\} \tag{2.3}$$
$$G_s^5 = \frac{\omega\Delta_0(\omega)(1-2\xi_s)}{1-\xi_s(\omega)}\left\{\gamma_s^2(\omega)[\tilde{\rho}-1+2\nu(\rho)-2n_s(\rho)]-4\Delta_0^2(\omega)h_s(\rho)\right\}$$
$$G_s^6 = \omega\Delta_0(\omega)\left\{2\frac{\mu_s}{\mu(\rho)}-\frac{\tilde{\rho}\gamma_s^2(\omega)-2\Delta_0^2(\omega)}{\gamma_s^2(\omega)(1-2\xi_s(\omega))}-1\right\}$$
$$G_s^7 = \omega\Delta_0(\omega)\eta_s(\omega)\left\{-2\frac{\mu_s}{\mu(\rho)}+\frac{\tilde{\rho}(\gamma_s^2(\omega)-2\Delta_0^2(\omega))}{\gamma_s^2(\omega)(1-\xi_s(\omega))}+1\right\}$$
$$G_s^8 = \frac{\omega\Delta_0(\omega)h_s(\omega)}{1-2\xi_s(\omega)}$$

Here

$$\xi_s(\omega)=\frac{\Delta_0^2(\omega)}{\gamma_s^2(\omega)},\quad \eta_s(\omega)=k_s^2(\omega)-\Delta_0^2(\omega),\quad h_s(\rho)=m_s(\rho)-1-n_s(\rho)+(2-m_s(\rho))\nu(\rho)$$
$$m_s(\rho)=\frac{\tilde{\rho}d^2(\rho)}{d_s^2},\quad n_s(\rho)=\frac{d_s^2}{\tilde{\rho}c^2(\rho)},\quad \nu(\rho)=\frac{d^2(\rho)}{c^2(\rho)},\quad \tilde{\rho}=\frac{\rho}{\rho_s}$$

It can be shown that the functions $\alpha_s^i = \alpha_s^i(z, \omega)$ satisfy the following differential equations

$$\frac{\partial^3\alpha_s^i}{\partial z^3}+4k_s^2(\omega)\frac{\partial\alpha_s^i}{\partial z}=0,\quad i=1,\ldots,4 \tag{2.4}$$
$$\frac{\partial^4\alpha_s^i}{\partial z^4}+2(k_s^2(\omega)+\gamma_s^2(\omega)-2\Delta_0^2(\omega))\frac{\partial^2\alpha_s^i}{\partial z}+(k_s^2(\omega)-\gamma_s^2(\omega))\alpha_s^i=0,\quad i=5,\ldots,8,$$
$$b_{s-1}\leqslant z\leqslant b_s,\quad s=1,\ldots,N$$

Suppose N^* is the optimal number of layers, $\rho_s^*(s = 1, \ldots, N^*)$ are the optimal physical parameters of the materials of the layers and $b_s^*(s = 1, \ldots, N^* - 1)$ are the optimal coordinates of the interfaces of the layers. The following condition is then satisfied for the optimum solution

$$R_s(f_s^{+*},\ f_s^{-*},\ p_s^*,\ q_s^*;\ \rho_s^*)|_z=\max_{\rho\in\Lambda} R_s(f_s^{+*},\ f_s^{-*},\ p_s^*,\ q_s^*;\ \rho)|_z, \tag{2.5}$$
$$b_{s-1}^*\leqslant z\leqslant b_s^*,\quad s=1,\ldots,N^*$$

We will now investigate the possibility of the existence of qualitative features of the structure of the optimum solutions in problems of optimum synthesis of the form (1.4), (1.5). We will consider the most interesting case both in its theoretical aspects as well as in its applied aspects when the required dependence $\tilde{T}(\omega)$ is such that, for each value of the frequency $\omega \in [\omega_{min}, \omega_{max}]$, the value of $\tilde{T}(\omega)$ is the limiting attainable value. The required dependence $\tilde{T}(\omega)$ can therefore take only two values separately: either zero (total reflection) or unity (total transmission).

The following classes of optimum synthesis problems belong to the type under consideration, in which it is required to ensure: maximum attentuation of an elastic wave in the specified range of frequencies $[\omega_{\min}, \omega_{\max}]$, minimum reflection of an elastic wave in the specified range of frequencies $[\omega_{\min}, \omega_{\max}]$, and maximum attenuation of an elastic wave in some regions of the spectrum and the minimum reflection in other regions.

3. INCLINED INCIDENCE OF A HARMONIC ELASTIC WAVE

We will consider the inclined incidence of a harmonic elastic wave of frequency ω on a system of elastic layers, where the permissible set consists of two materials. In the case being considered, $\rho_s = \rho_{s-2}$, $c_s = c_{s-2}$, $d_s = d_{s-2}$, $s = 3, \ldots, N$, that is, the materials of all the even and odd layers will have identical physical properties. When account is taken of the structure of an equation of the form (2.4) for the functions $\alpha_s^l(z, \omega)$, $b_{s-1} \leq z \leq b_s (s = 1, \ldots, N)$, occurring in the functions $R_s(f_s^+, f_s^-, p_s, q_s; \rho)$ (2.1) and, also, the properties of the solutions of the initial system (1.4), it is possible to find the connection between the functions $\alpha_s^l(z, \omega)$ for the layers with numbers s and $s + 2$ (they possess the same physical properties). Such a constructive analysis enables us to establish that the following equalities hold for the optimum solution

$$R_{s-2}(f_s^{+*}, f_s^{-*}, p_s^*, q_s^*; \rho)|_z = R_s(f_s^{+*}, f_s^{-*}, p_s^*, q_s^*; \rho)|_z \tag{3.1}$$

$$b_{s-3}^* \leq z \leq b_{s-2}^*, \quad s = 4, \ldots, N^* - 1$$

Since, for the optimum solution, the functions R_s for the inner layers with the same physical properties have the same structure, the distance between the optimum coordinates of the interfaces of the layers are singular points for the functions R_s since, at these points, the maximum value of the functions R_s is simultaneously attained for the different elements of the set Λ. From inequality (3.1), we therefore immediately obtain

$$\Delta_s^* = \Delta_{s-2}^*, \quad s = 4, \ldots, N^* - 1 \tag{3.2}$$

Hence, it has been established that the following assertion holds.

Assertion 1. Suppose that, in the inclined incidence of a harmonic elastic wave on a system of elastic layers, the permissible set consists of just two materials. Then, in the optimum structure, the thickness of the inner layers with identical physical properties is the same.

This property of inner symmetry in optimum structures enables us to reduce the dimensionality of the initial synthesis problem considerably and to reduce the multidimensional synthesis problem, the dimensionality of which is determined by the overall number of layers of the optimum structure, to a three-parameter problem. The three independently variable parameters are the thicknesses of the inner layers with different physical properties and the thickness of one of the boundary layers. Consequently, in the case under consideration, the complete set of parameters which realize the limiting possibilities of composite structures can be effectively distinguished.

4. INCLINED INCIDENCE OF A NON-MONOCHROMATIC ELASTIC WAVE

We will now consider the general case of the inclined incidence of a non-monochromatic elastic wave on a system of elastic layers. We will consider the optimum synthesis problem (1.4), (1.5) with the following additional assumption: at a certain frequency $\omega = \omega^* \in [\omega_{\min}, \omega_{\max}]$, the value of the energy transmission coefficient of the structure being designed $T(\omega)$ must be the limiting possible value, that is,

$$T(\omega^*) = T_{\omega^*}^*(\omega^*) \tag{4.1}$$

(the right-hand side of the equality is the limiting attainable value of the energy transmission coefficient at a frequency $\omega = \omega^*$).

Since the greater the width of the spectrum, the higher the approximation, the value of the energy transmission coefficient will be closest to the required value at a frequency $\omega = \omega^*$ for the case of harmonic interaction with this frequency. The set of globally optimum solutions in the optimum synthesis problem (1.4), (1.5) with an additional condition of the form of (4.1) will therefore be a subset of the globally optimum solutions for the case of harmonic interaction with a frequency $\omega = \omega^*$. Consequently, in the case being considered, the structure of the optimum solutions is qualitatively the same as for the case of harmonic interactions.

For the more general case of the inclined incidence of elastic waves on a system of elastic layers, the relation between the parameters in the structures, which provide the limiting possibilities of the structure, therefore also possesses the same qualitative features and, in particular, the same type of inner symmetry as for the cases of electromagnetic and acoustic waves considered earlier [1].

5. INCLINED INCIDENCE OF AN ELASTIC WAVE CLOSE TO NORMAL INCIDENCE

We will now consider the question of the relation between the qualitative features of the structure of the optimum solutions for the cases of normal and close to normal incidence of an elastic wave.

We expand the solutions of initial system (1.4) and the system conjugate to it in powers of a small parameter ϑ_0. The functions R_s (2.1) can then be represented as

$$R_s(f_s^+,\ f_s^-,\ p_s,\ q_s;\ \rho)|_z = \int_{\omega_{\min}}^{\omega_{\max}} \left[\frac{1}{\rho c^2(\rho)} \beta_s^1(z,\ \omega) + \rho \beta_s^2(z,\ \omega) \right] d\omega + O(\vartheta_0^2) \tag{5.1}$$

$$b_{s-1} \leqslant z \leqslant b_s,\quad s = 1, \ldots, N$$

The functions $\beta_s^1(z, \omega)$ and $\beta_s^2(z, \omega)$ are expressed in terms of the solutions of initial system (1.4) and the system conjugate to it.

Hence, for cases of close-to-normal incidence of an elastic wave, the structure of the functions R_s (5.1) is analogous to the structure of the functions R_s for the case of normal incidence of elastic waves [1]. This means that, for these cases, the same qualitative features of the structure of the optimum solutions as in the normal incidence of a non-monochromatic elastic wave [1] can be established.

A constructive analysis of the necessary conditions of optimality, carried out by analogy with the results obtained previously [1], enable us to establish the following.

We introduce the function

$$\varphi(\rho,\ \tau) = \frac{\mu(\rho)}{\rho} + \tau\rho,\quad -\infty < \tau < +\infty;\quad \mu(\rho) = \frac{1}{c^2(\rho)} - \frac{\sin^2\vartheta_0}{c_0^2}$$

and use the notation

$$\rho^+(\tau) = \arg\max_{\rho\in\Lambda} \varphi(\rho,\ \tau),\quad \rho^-(\tau) = \arg\min_{\rho\in\Lambda} \varphi(\rho,\ \tau)$$

$$(x^* = \arg\operatorname*{extr}_x A(x),\ \text{ if }\ A(x^*) = \operatorname*{extr}_x A(x))$$

($\rho^+(\tau)$ is a monotonically increasing function and $\rho^-(\tau)$ is a monotonically decreasing function of the argument τ and, by virtue of the discreteness of the set Λ, both of these functions are piecewise-constant). Only materials with properties which belong to the domains of values of these functions can occur in the optimum elastic system. Analysis of the functions $\rho^+(\tau)$ and $\rho^-(\tau)$ therefore gives considerable information on the structure of the optimum construction.

Assertion 2. In the case of close-to-normal incidence of an elastic wave on a system of elastic layers, the number of different materials constituting the optimum elastic system cannot exceed $p + q$, where p and q are the numbers of points of discontinuity of the functions $\rho^+(\tau)$ and $\rho^-(\tau)$ respectively.

In the elements of the set Λ, we introduce the function

$$L(\alpha,\ \beta) = -\frac{\mu(\beta) - \mu(\alpha)}{\beta - \alpha},\quad \alpha,\ \beta \in \Lambda$$

We shall consider the materials occurring in the set Λ to be arranged in order of increasing densities.

Assertion 3. In the case of close-to-normal incidence of an elastic wave, the physical properties of the materials forming the optimum elastic system satisfy the following system of recurrence relations

$$L(\rho^{j_{r-1}^{+}},\ \rho^{j_r^{+}}) = \min L(\rho^{j_{r-1}^{+}},\ \rho),\ \rho^{j_{r-1}^{+}} \leqslant \rho \leqslant \rho_{\max}$$
$$r = 1, \ldots, p;\quad j_0^{+} = 1,\quad j_p^{+} = m \tag{5.2}$$

$$L(\rho^{j_{r-1}^{-}},\ \rho^{j_r^{-}}) = \max L(\rho^{j_{r-1}^{-}},\ \rho),\ \rho^{j_{r-1}^{-}} < \rho \leqslant \rho_{\max}$$
$$r = 1, \ldots, q;\quad j_0^{-} = 1,\quad j_q^{-} = m \tag{5.3}$$

For close-to-normal incidence of an elastic wave, the physical properties of the materials forming the optimum elastic system can only be elements of the set $\Lambda^* = \Lambda^+ \cup \Lambda^-$, where

$$\Lambda^{\pm} = \{\rho_{\min} = \rho^{j_0^{\pm}} < \rho^{j_1^{\pm}} < \ldots < \rho^{j_p^{\pm}} = \rho_{\max}\}$$

For close-to-normal incidence of an elastic wave, the system of recurrence relations (5.2), (5.3) enables us to carry out an effective, a priori reduction of the set of permissible versions Λ. Here, new possibilities are revealed for predicting the physical properties of materials, the inclusion of which in the initial set leads to a significant improvement in the functional characteristics of the structure being designed.

Assertion 4. In the case of close-to-normal incidence of an elastic wave, the physical parameters of the materials of neighbouring layers of the optimum elastic system are adjacent in the sequence

$$\ldots\rho^{j_0^{+}},\ \ \rho^{j_1^{+}},\ \ \ldots\rho^{j_p^{+}},\ \rho^{j_q^{-}},\ \ \rho^{j_{q-1}^{-}},\ \ \ldots\rho^{j_0^{-}},\ \ \rho^{j_0^{+}},\ \ \rho^{j_1^{+}},\ldots \tag{5.4}$$

This property establishes the nature of the junction of the materials of the layers with different physical properties for the optimum solution. Inhomogeneous structures, for which the order of alternation of the materials of the layers differs from (5.4), are not optimum.

The established specific features of the structure of optimum elastic systems enables us to reduce considerably the number of permissible versions of multilayer constructions which are analysed for optimality. Knowledge of these features enables us to increase the efficiency of the different methods for searching for the optimum solution and to broaden the limits of applicability of the different approaches.

6. THE CONDITIONS UNDER WHICH NO MORE THAN TWO MATERIALS CAN FORM THE OPTIMUM ELASTIC SYSTEM

We will now consider the case of a harmonic elastic wave.

It has been established in [3] that, if a laminar structure is composed of materials in which shear waves do not occur, then the optimum construction can consist of no more than two materials from the permissible set, regardless of the number of materials constituting the initial set. A constructive analysis of the necessary conditions of optimality, carried out by analogy with the case when shear waves do not occur in the layers [3], enables us to establish that, under certain conditions, this property can also hold in the more general case when shear waves can occur in the layers.

Assertion 5. At sufficiently small angles of incidence of a harmonic wave on a system of elastic layers, regardless of the number of materials constituting the initial set, the optimum laminar structure will consist of no more than two materials from the permissible set, the physical properties of which are adjacent in sequence (5.4).

It is obvious that the property which has been formulated also holds in the case of non-monochromatic elastic waves but with the additional assumption, formulated in Section 4, that, at a certain frequency $\omega^* \in [\omega_{\min}, \omega_{\max}]$, the value of the energy transmission coefficient $T(\omega)$ of the structure being designed must be the limiting attainable value, that is, with the additional satisfaction of condition (4.1).

Hence, for the cases considered above, the optimum synthesis problem which has been formulated can be completely solves, that is, the complete set of versions of laminar structures which realize the limiting possibilities can be distinguished.

We can conclude from the above analysis that the necessary conditions of optimality (2.5) contain substantial information on the structure of the optimum construction and, despite the complicated form of the Hamilton functions for problem (2.1)–(2.3) the qualitative structure of the optimum solution can be successfully revealed by a constructive analysis.

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